18/11/2018

NUMERICAL SOLUTION OF ORDER

- Equation

i - Taylor's series method; if you have a function f(x) with a continuous derivatives at the point $x=a$ then, The taylor's series expansion of the function $x=a$ is given by

$$f(x+a) = f(a) + (x-a)f'(a) + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 f'''(a)}{3!} + \cdots + \frac{(x-a)^n f^n(a)}{n!} + E_{n+1}$$

where $E_{n+1} \leq \left| \frac{(x-a)^{n+1} f^{n+1}(\varepsilon)}{(n+1)!} \right|$ $\varepsilon \in (a, x)$ $n \in \mathbb{N}$

if $a=0$ Then

The series is called maclaurin's series

if $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

Then taylor's series expansion is given as

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2 y''(x_0)}{2!} + \frac{(x-x_0)^3 y'''(x_0)}{3!} + \cdots + \frac{(x-x_0)^n y^n(x_0)}{n!} + E_{n+1}$$

$$E_{n+1} \leq \left| \frac{(x-x_0)^{n+1} f^{n+1}(\varepsilon)}{(n+1)!} \right| \quad \varepsilon \in (x_0, x)$$

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Initial where problem

eg: $\frac{dy}{dx} = xy \quad y(0) = 1 \quad y'(0) = 5$

Bounding value problem

$y'' - 2y' + 4x^2 y = 0 \quad$ at $y'(1) = 0$

$y(3) = -1$

Because the value of the solution it know between two different point.

$y'(x_0), y''(x_0), y'''(x_0)$ ---

Can be obtained as following

$y' = f(x, y)$

$y'(x_0) = f(x_0, y_0)$

$y''(x_0) = f'(x_0, y_0')$

$y'''(x_0) = f''(x_0, y_0'')$

$\vdots$

$y^n(x_0) = f^{n-1}(x_0, y_0^{n-1})$

Use the taylor series to find the first fourth term of ~~eq~~

$$\frac{dy}{dx} + xy = e^x \qquad y(0) = 1$$

$$\frac{dy}{dx} = e^x - xy$$

$x_0 = 0 \qquad y_0 = 1$

$$y(x) = y(x_0) + xy'(x_0) + \frac{x^2 y''(x_0)}{2!} + \frac{x^3 y'''(x_0)}{3!} + $$

$\frac{x^4 y^{iv}(x_0)}{4!} + E_{n+1}$

$+E_5$

$y'(x_0) = y'(0) = e^0 - 0.1 = 1$

$y''(x_0) = e^0 - 1 - 0.1 - 1 - 1 - 1 = 0 \quad y'' = e^x - y - xy'$

$y'''(0) = 1 - 2.1 - 0.0 = -1$

$y^{iv}(0) = 1 - 3.0 - 0 = 1 \quad y''' = e^x - 2y'' - xy''$

$y(x) = 1 + x - \frac{x^3}{6} + \frac{x^4}{24}$

$E_5 \leq \left| \frac{x^3 f^v(\varepsilon)}{5!} \right| \quad \varepsilon \in (0, x)$

let $\varepsilon = 0 \quad y^v = e^x - 4y''' - xy^{iv}$

$E_5 = \left| \frac{5x^5}{5!} \right| \quad y(0) = 1 + 4 = 5$

$= \left| \frac{5x^5}{5 \times 4!} \right| = \left| \frac{x^5}{24} \right|$

$E_5 = \left| \frac{x^5}{24} \right|$

At $x = 1.5$

$\frac{dy}{dx} + xy = e^x$

use taylor be to solve the initial problem

$y'' = y' + xy^2$

$y(0) = 1 \quad y'(0) = -1$

$y'' = y + xy' \quad y(0) = 1 \quad , \quad y'(0) = -1$

$y(x) = y(0) + xy'(0) + \frac{xy''(0)}{2!} + \frac{x^3 y'''(0)}{3!} +$

$\frac{x^4 y^{iv}(0)}{4!} + E_5$

$y''(0) = -1 + 0 \cdot (1)^2 = -1 \quad , \quad y'''(0) = -1 + 1 + 0 = 0$

$y^{iv}(0) = 0 - 4 = -4 \quad y(x) = 1 - x - \frac{x^2}{2} - \frac{x^4}{6}$

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$y''' = y'' + y^2 + 2xy'$

$y^{iv} = y''' + 2y' + 2y' + 2xy''$

Picards Method

If $y' = f(x, y)$, $y(x_0) = y_0$

Now integrating both sides, we have

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx$$

$$\Rightarrow y(x)\Big|_{y_0}^{y} = \int_{x_0}^{x} f(t, y(t))\, dt$$

$$y - y_0 = \int_{x_0}^{x} f(t, y(t))\, dt$$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

$$\therefore y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt \text{ ----- } *$$

$*$ is called the picard's iterative (or polynomials) $y_1, y_2, y_3, \ldots$ named after the french Mathematician prof. Emile Picard

Example

Use the picard method to determine (up to the 4th power) the solution of the I.V.P $y' = 2xy$, $y(0) = 1$

Solution

① $f(x, y) = 2xy$ ② $x_0 = 0$ $y_0 = 1$

$$y_1(x) = y_0 + \int_0^x f(t, y_0(t))\, dt$$

$$= 1 + \int_0^x 2t(1)\, dt$$

$$= 1 + \int_0^x 2t\, dt = 1 + t^2\Big|_0^x = \underline{\underline{1 + x^2}}$$

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$$y_2(x) = y_0 + \int_0^x f(t, y_1(x))\,dt$$
$$1 + \int_0^x 2t(1+t^2)\,dt$$
$$1 + \int_0^x (2t + 2t^3)\,dt$$
$$= 1 + \left[t^2 + \frac{t^4}{2}\right]\Big|_0^x$$
$$y_2(x) = 1 + x^2 + \frac{x^4}{2}$$
$$y_3(x) = 1 + \int_0^x \left(1 + t^2 + \frac{t^4}{2}\right)dt = 1 + t + \frac{t^3}{3} + \frac{t^5}{10}\Big|_0^x$$
$$y_3(x) = 1 + x + \frac{x^3}{3} + \frac{x^5}{10}$$

(2) $y' = 3x - y^2$, $y(1) = 2$.

$f(x,y) = 3x - y^2$ $x_0 = 1$, $y_0 = 2$.

$$y_1(x) = y_0 + \int_1^x f(t, y_0(t))\,dt$$
$$= 2 + \int_1^x (3t - 4)\,dt$$
$$= 2 + \left[\frac{3t^2}{2} - 4t\right]_1^x$$
$$= 2 + \frac{3x^2}{2} - 4x - \frac{3}{2} + 4$$
$$y_1(x) = \frac{3x^2}{2} - 4x + 9/2$$
$$y_2(x) = y_0 + \int_1^x f(t, y_1(t))\,dt$$
$$= 2 + \int_1^x \left(3t - \left[\frac{3t^2}{2} - 4t + \frac{9}{2}\right]^2\right)dt$$
$$2 + \int_1^x \left[3t - \left[\frac{81}{4} - \frac{2 \times 9}{2}\left(4t - \frac{3}{2}t^2\right) + \left(4t - \frac{3}{2}t^2\right)^2\right]\right]dt$$
$$2 + \int_1^x \left[3t - \frac{81}{4} - 36t + 22t\right.$$
$$y_2(x) = 2 + \frac{9}{20}x^5 - 3x + \frac{59x^3}{6} - \frac{33}{2}x^2 - \frac{81}{4}x - \frac{9}{20} +$$

$$3 - \frac{59}{6} + \frac{33}{2} + \frac{81}{4}$$

EXERCISE

1. Use Taylor's Series method to evaluate $y(1.8)$ from the I.V.P $y'' = y' + xy^2$, $y(1) = 0$, $y'(1) = 0.5$ upto 4th power

(2) Use Picards method to evaluate (up to x^5) $y(1.6)$ from the I.V.P $y' = xy + 2x - 3$, $y(1) = 0$. Hence estimate that actual (absolute error by comparing with the exact solution

Unit step and multistep method.

Def: A numerical sheme as called a unstep method if the value of the function is computed from the previous value is a previous step while in a multistep method, the value of the function is computed from more than one (previous) step

Example: of unistep (one - step) methods include
(i) Euler method (ii) Improved Euler (iii) R.K method

multistep

Example of multistep method include milne thompson
(i) Predictor - corrector method

Adams Barh-firth

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① Euler method: This method is named after the ~~Swiss~~ swiss Mathematician leunhard Euler (1707, 178

Given the IVP $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$.

if x have the values $x_0, x_0+h, x_0+2h + x_0+3h, ---$

$x_0 + nh$ or $x_0, x_1, x_2 - - - - x_n$.

corresponding to the value of $y = y_0, y_1, y_2, y_3,$

y_n and h is the step size, then.

$y_1 = y_0 + hf(x_0, y_0)$, $y_2 = y_1 + hf(x_1, y_1)$

$y_3 = y_2 + hf(x_2, y_2)$, $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

Proof

from the taylor's series of $y(x)$ that is

$y(x) = y(x_0) + (x - x_0)y'(x_0) + E(0)$.

$\Rightarrow y'(x_0) = \frac{y(x) - y(x_0)}{x - x_0}$.

but $x - x_0 = h$ and $y'(x) = f(x, y) \Rightarrow$

$f(x, y) = \frac{y(x) - y(x_0)}{h}$

$\Rightarrow y(x) - y(x_0) = hf(x, y)$.

$\Rightarrow y(x) = y(x_0) + hf(x, y)$.

$y_1(x) = y(x_0) + hf(x_0, y_0)$.

Example

Solve by Euler's method $y' = x + y$, $y(0) = 1$

$y(0.3)$, $h = 0.1$

Solution

① $f(x,y) = x+y$ ② $y(0) = 1 \Rightarrow x_0 = 0,\ y_0 =$

③ $h = 0.1$ $x_1 = x_0 + h$

$0 + 0.1$

$y_1 = y_0 + hf(x_0, y_0)$ $x_1 = 0.1$

$= 1 + 0.1(0+1) = 1.1$

$x_1 = 0.1,\ y_1 = 1.1$

$y_2 = y_1 + hf(x_1, y_1)$

$y_2 = 1.1 + 0.1\,(x_1, y_1)$

$= 1.1 + 0.1\,(0.1 + 1.1)$

$y(0.2) = y_2 = 1.22$

$x_2 = 0.2,\ y_2 = 1.22$

$y_3 = y_2 + hf(x_2, y_2)$

$= 1.22 + 0.1\,(0.2 + 1.22)$

$y(0.3) = y_3 = 1.363$

$y' = x + y$

$y' - y = x$

$P(x) = -1$ $Q(x) = x$.

$\text{If } e^{-\int dx} = e^{-x}$

$y\,\text{If} = \int \text{IF}\,Q\,dx + C$

$y e^{-x} = \int x e^{-x}\,dx$

$y = e^{x}\left[\int x e^{-x}\,dx + C\right]$ evaluate $x = 0.1$

$x = 0.2$

$x = 0.3$.

Example 2

Use Euler's method with step size 0.3 to compute the approximation y-value $y(0.9)$ of the solution of I.V.P $y' = x^2$; $y(0) = 1$

Solution

① $f(x, y) = x^2$ ② $y(0) = 1$, $x_0 = 0$, $y_0 = 1$

③ $h = 0.3$

$y_1 = y_0 + hf(x_0, y_0)$

$= 1 + 0.3(x_0^2) = 1$

$x_1 = 0.3$; $y_1 = 1$

$y_2 = y_1 + hf(x_1, y_1)$

$y_2 = 1 + 0.3(x_1^2)$

$= 1 + 0.3(0.3)^2$

$x_2 = 0.6$; $y_2 = 1.027$

$y_3 = y_2 + hf(x_2, y_2)$

$= 1.027 + 0.3(x_2)^2$

$= 1.027 + 0.3(0.6)^2$

$= 1.135$ //

Example 3

$y' = x + 2y$ $h = 0.1$ $y(0) = 0$, $y(0.4)$.

① $f(x, y) = x + 2y$ ② $y(0) = 0$ $x_0 = 0$, $y_0 = 0$

$h = 0.1$ $h = 0.2$

$y_1 = y_0 + hf(x_0, y_0) = 0 + 0.1[x_0 + 2 \times y_0)$.

$x_4 = \quad y_4$

$y(x=4) = y_4$

$0 + 0.1(0 + 2 \times 0)$

$y(0.1) = y_1 = 0$

$x_1 = 0.1, \quad y_1 = 0$

$y_2 = y_1 + hf(x_1 y_1)$

$0 + 0.1(x_1 + 2 \times y_1)$

$y_2 = 0 + 0.1(0.1 + 2 \times 0)$

$y(0.2) = y_2 = 0.01$

$x_2 = 0.2, \quad y_2 = 0.01$

$y_3 = y_2 + hf(x_2, y_2)$

$= 0.01 + 0.1(0.2 + 0.01)$

$= 0.01 + 0.1(0.21)$

$\cancel{y(0.3)} \quad y_3 = 0.031$

$x_3 = 0.3$

$y_4 = y_3 + hf(x_3, y_3)$

$= 0.031 + 0.1(0.3 + 0.031)$

$\Rightarrow x_4 = 0.4 \quad y_4 = 0.0641$

Improved Euler's method

The above scheme can be improved upon by the scheme

$$z_1 = y_0 + hf(x_0, y_0) \Rightarrow z_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, z_1)].$$

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1\, y_1^{(0)})]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$y_n = y_{n-1} + \frac{h}{2}[f(x_{n-1}, y_{n-1}) + f(x_n, z_n)$$

Example

$y' = xy$, $y(1) = 1$ and $h = 0.1$

approximate $y(1.3)$ using improved Eulors method.

Solution

$$z_1 = y_0 + hf(x - x_0)$$

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, z_1)]$$

① $f(x, y) = xy$, ② $y(1) = 1 \Rightarrow x_0 = 1$ $y_0 = 1$

③ $h = 0.1$

$$z_1 = 1 + 0.1\,[1 \times 1] = 1.1$$

$$y_1 = 1 + \frac{0.1}{2}[1 \times 1 + 1.1 \times 1.1]$$

$$= 1 + 0.05\,[2.21] \Rightarrow y_1 = 1.1105$$

$x_1 = 1.1$ and $y_1 = 1.1105$

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$$y_2 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, z_2)\right]$$

Ex

Approximate $y(0.6)$ for each of the i.v.p below using the improved Euler's method with a step size of $h = 0.2$.

① $y' = x - y$, $y(0) = 2$

② $y' = \frac{x^2 - 2xy}{x^2 + 2y^2}$ $y(0) = -1$

③ $y' = 5y$, $y(0) = 1$

④ $y' = \frac{x - y}{x + 2y}$ $y(0) = 1$

Also obtain their exact solution and evaluate $y(0.6)$. Compare your result with the improved Euler's method

RUNGE – KUTTA

The Runge–Kutta method was developed by Carl Runge (1826 – 1927) and Kutta (1861 – 1944) both German mathematicians. It has an advantage over the taylor's series in the sense that it does not use derivatives. It has a higher degree of accuracy than the previous methods and as such is the most widely used.

if $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

Then $y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

where $k_1 = hf(x_0, y_0)$, $k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1)$.

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2)$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.

$k_1 = hf(x_1, y_1)$, $k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_1)$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_2)$.

$k_4 = hf(x_1 + h, y_1 + k_3$.

⋮

$y_n = y_{n-1} + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$.

$k_1 = hf(x_{n-1}, y_{n-1})$, $k_2 = hf(x_{n-1} + \frac{h}{2}, y_{n-1} + \frac{1}{2}k_1)$.

$k_3 = hf(x_{n-1} + \frac{h}{2}, y_{n-1} + \frac{1}{2}k_2)$

$k_4 = hf(x_{n-1} + h, y_{n+1} + k_3)$

Example

find $y(0.1)$ and $y(0.2)$ from the I.V.P

$y' = x^2 + y$, $y(0) = 1$, $h = 0.1$

Solution

① $f(x, y) = x^2 + y$. ② $y(0) = 1 \Rightarrow x_0 = 0$, $y_0 = 1$

③ $h = 0.1$

$k_1 = hf(x_0, y_0)$

$k_1 = 0.1(x_0, y_0) = 0.1[0^2 + 1] = 0 + 0.1 \Rightarrow k_1 = 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right)$

$= 0.1\left(0 + 0.05, 1 + 0.05\right)$

$= 0.1\left[0.05, 1.05\right)$

$k_2 = 0.1\left(0.05^2 + 1.05\right) = 0.10525$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right)$

$= 0.1f\left(0.05, 1.05262\right)$

$= 0.1\left(0.05^2 + 1.05262\right)$

$= 0.10551$

$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$

$= 0.1f\left(0.1, 1.1055\right)$

$= 0.1\left[0.1^2 + 1.10551\right]$

$= 0.11155$

$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$= 1 + \frac{1}{6} + \left(2 \times 0.10525\right) + \left(2 \times 0.10551\right) + 0.11155\right)$

$y_1 = 1.10551$

$y_2 = 1.224205$

$y(0.2) = 3e^{0.2} -$

$y(x) = 3e^x - x^2 - 2x - 2$

$y(0.2) = 1.224208$

Error $= 1.22408 - 1.224205$

$= 0.000003$

30-01-2018

MILNE THOMPSON

The method of Taylor's Euler and Rungi kuatta are called single – step methods because they use only the information from one previous point to compute the successive point that is only the initial point (x_0, y_0) is used to compute (x_1, y_1) and in general y_n is needed to compute y_{n+1}. After several points have been found it is feasible to use several prive points in the calculation, The Milne – Thompson method used y_{n-3}, y_{n-2}, y_{n-1} and y_n in the calculation of y_{n+1}. This method is not self-starting four initial points $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$ must be given in advence in order generate the point $\{(x_n, y_n)\}$

If $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$ then $y_4^p = y_0 + \frac{4}{3} h(2f_1 - f_2 + 2f_3)$ and $y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$

Note that the formular above can be derived from the Milne – integration, that is

$\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, integrating both sides w.r.t x from x_0 to $x_0 + 4h$ given

$$\int_{x_0}^{x_0+4h} dy = \int_{x_0}^{x_0+4h} f(x,y)\,dx \qquad \begin{cases} x = x_0 + hp \\ dx = h\,dp \\ \Rightarrow p = 4 \end{cases}$$

$$y_4 - y_0 = h\int_0^4 f(p)\,dp$$

expanding $f(p)$ using the Newtons interpolation difference formular of degree 3 we have

$$y_4 - y_0 = h\int_0^4 f_0 + p\Delta f_0 + \frac{p(p-1)}{2}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 f_0$$

$$= h\left[pf_0 + \frac{p^2}{2}\Delta f_0 + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 f_0 + \frac{\left(\frac{p^4}{4} - p^3 + p^2\right)}{6}\Delta^3 f_0\right]$$

$$= h\left[4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0\right]$$

Now replacing this finite difference with the functional values we have

$\Delta f_0 = f_1 - f_0$, $\Delta^2 f_0 = f_2 - 2f_1 + f_0$, $\Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0$

$$\therefore y_4 - y_0 = h\left[4f_0 + 8(f_1 - f_0) + \frac{20}{3}(f_2 - 2f_1 + f_0) + \frac{8}{3}(f_3 - 3f_2 + 3f_1 - f_0)\right]$$

$$y_4 - y_0 = h\left[\frac{8}{3}f_1 - \frac{4}{3}f_2 + \frac{8}{3}f_3\right]$$

$$\therefore y_4^p = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

Similarly, integrating the ivp from $x_0 + 2h$ to $x_0 + 4h$ the corrector formular can be obtained

Example

use the Milne - Thompson method to determine $y(0.4)$ and $y(0.5)$ from the I.V.P

$\frac{dy}{dx} = x^2 + y$, $y(0) = 1$ taking $h = 0.1$

The taylors series method to determine the starting values

Solution

$y(x) = y_0 + xy'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0)$.

① $\frac{dy}{dx} = x^2 + y$ ② $y(0) = 1$ ③ $h = 0.1$.

$y(0) = 1$, $x_0 = 0$

$y'(0) = 0^2 + 1 = 1$

$y''(0) = 2x + y' = 2(0) + 1 = 1$

$y'''(0) = 2 + y''(0) = 2 + 1 = 3$.

$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{2}$

$x_1 = 0.1$ $y_1(0.1) = 1 + 0.1 + \frac{0.1^2}{2} + \frac{0.1^3}{2} = 1.1055$

$x_2 = 0.2$, $y_2(0.2) = 1 + 0.2 + \frac{0.2^2}{2} + \frac{0.2^3}{2}$

$= 1.2280$

$x_3 = 0.3$

$y_3(0.3) = 1 + 0.3 + \frac{0.3^2}{2} + \frac{0.3^3}{2} = 1.3585$

$x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$

$y_0 = 1$, $y_1 = 1.1055$, $y_2 = 1.2280$, $y_3 = 1.3585$

$y_4^P = y_0 + \frac{4}{3} h [2f_1 - f_2 + 2f_3]$

$f_1 = f(x_1, y_1) = x_1^2 + y_1 = 0.1^2 + 1.1055 = 1.1155$

$f_2 = f(x_2, y_2) = x_2^2 + y_2 = 0.2 + 1.2280 = 1.2680$

$f_3 = f(x_3, y_3) = x_3^2 + y_3 = 0.3 + 1.3585 = 1.4485$

$y_4^P = 1 + \frac{4 \times 0.1}{3}\left[2 \times 1.1155 - 1.2680 + 2 \times 1.4483\right]$

$= 1.5152.$

$f_4 = f(x_4, y_4^P) = x_4^2 + y_4^P = 0.4^2 + 1.5152.$

$= 1.6752.$

$y_4^C = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$

$= 1.2240 + \frac{0.1}{3}\left[1.2680 + 4 \times 1.4485 + 1.6752\right]$

$= 1.2240 + \frac{0.1}{3}\left[8.7332\right]$

$= \underline{1.5151}$

$\frac{dy}{dx} = x^2 + y$

$\frac{dy}{dx} - y = x^2$

$IF = e^{\int -dx} = e^{-x}$

$\int \frac{d}{dx}\left[y\, IF\right] dx = \int e^{-x} x^2 dx.$

$y e^{-x} = \int x^2 e^{-x} dx$

$y = 4e^{4x} - x^2 - 2x - 2.$

$y(0.4) = 3 \times e^{4 \times 0.4} - 0.4^2 - 2 \times 0.4 - 2.$

$= 1.51547$

Error $= 1.51547 - 1.5151$

$$y_4^p = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$y_4^c = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$$

$$y_5^p = y_1 + \frac{4h}{3}\left[2f_2 - f_3 + 2f_4\right]$$

$$y_5^c = y_3 + \frac{h}{3}\left[f_3 + 4f_4 + f_5\right]$$

Assignment y_5^c

$$y_k^p = y_{k-4} + \frac{4h}{3}\left[2f_{k-3} - f_{k-2} + 2f_{k-1}\right]$$

$$y_k^c = y_{k-2} + \frac{h}{3}\left[f_{k-2} + 4f_{k-1} + f_k\right]$$

$k \geq 4 \in \mathbb{N}$

Adams Bashforth and Moulton Predictor - Corrector formular

if $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ is an IVP

Then $y_4^p = y_0 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$ - (Adams Bashforth)

$y_4^c = y_3 + \frac{h}{24}\left(9f_4 + 19f_3 - 5f_2 + f_1\right)$ - (Adams - Moulton).

The Adams - Bashforth predictor formular can be obtained from the Newton's Backward difference formular that is, integrate the IVP from x_0 to $x_0 + h$ given

$$\int_{x_0}^{x_0+h} dy = \int_{x_0}^{x_0+h} f(x,y)\,dx$$

$$\Rightarrow y_1 - y_0 = h\int_{x_0}^{x_0+h} f(x,y)\,dx$$

$$y_1 - y_0 = h\int_0^1 f(p)\,dp = hf'f_0 + p\nabla f_0 + p(p+1)\nabla^2 f_0$$

Example

Use the Adams-Bashforth predictor-corrector formular to evaluate $y(0.4)$ and $y(0.5)$ from the IVP $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$

$y(0.5)$ $y_4^p = 1.1556$ $y_4^c = 1.5008$

$y(0.4)$, $y(0.5)$ $h = 0.1$

06-02-2018

SYSTEM OF LINEAR EQUATIONS

A system of $m \times n$ linear eqn is a system of the form $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

These system of linear equation is very important as quite often problem's in since, engineering, Economics, etc lead to the these systems eg the analysis of electric circuits consist of invariant elements, analysis of networks etc —

i:- The solut^n of these systems can be classified in to direct method (exact) these include.

i) Cramer's rule

ii) matrix inverse method

iii) Gaussion elimanation method

② i:- The numerical iterate: these include Jacobi method

ii) Gauss seidel method also knwn as Lieb main method of successive displacement

iii) successive over-relaxation method

Cramer's rule

If $|A| = \Delta$, then $x_i = \frac{\Delta_i}{\Delta}$ where Δ_i is the determinant of the matrix obtained by replacing the ith column with the constant

Drawback: i:- The above system is difficult to apply if the order of the matrix $(n) > 4$

ii) $\Delta = 0$

- matrix inverse method

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A^{-1}B \qquad A^{-1} = \frac{\text{Adj}A}{|A|}$$

Drawback if $\Delta = 0$ the process again fails

GAUSSIAN ELIMANATION

This method named after the German, Mathematician, Elimination Gaussian) is the mostly widely applied as it does not require evaluation of the determinant. The system can be also be applied with non-square matrices most importantly the method can be implemented (easily) use computer.

Drawback: The system is cumbersome and more difficult to apply if the coefficient $[a_{ij}]$ are irrational numbers etc.

Example:

$2x_1 + x_2 - x_3 + 2x_4 = 5$

$4x_1 + 5x_2 - 3x_3 + 6x_4 = 9$

$-2x_1 + 5x_2 - 2x_3 + 6x_4 = 4$

$4x_1 + 11x_2 - 4x_3 + 8x_4 = 2$

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 4 & 5 & -3 & 6 & 9 \\ -2 & 5 & -2 & 6 & 4 \\ 4 & 11 & -4 & 8 & 2 \end{array}\right] \quad \begin{array}{l} R_2 \Rightarrow 2R_1 - R_2 \\ R_3 \Rightarrow R_1 + R_3 \\ R_4 \Rightarrow 2R_1 - R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 0 & -1 & 4 & 9 \\ 0 & 0 & 1 & -2 & -8 \end{array}\right] \quad \begin{array}{l} R_3 \Rightarrow 2R_2 - R_3 \\ R_4 \Rightarrow 3R_2 - R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & & & & \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 0 & -1 & 4 & 11 \\ 0 & 0 & 1 & -2 & -5 \end{array}\right] \quad R_4 \Rightarrow R_3 + R_4$$

$$\begin{bmatrix} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 0 & -1 & 4 & 11 \\ 0 & 0 & 0 & +2 & 6 \end{bmatrix}$$

$2x_1 + x_2 - x_3 + 2x_4 = 5$

$3x_2 - x_3 + 2x_4 = -1$

$-x_3 + 4x_4 = 11$

$2x_4 = 6$

$x_4 = \frac{6}{2} = 3 \Rightarrow x_4 = 3$

$2x_2 = -6 \Rightarrow x_2 = -3.$

$-x_3 + 4 \times 3 = 11$

$-x_4 + 12 - 11 \Rightarrow -x_3 + 1 = 0 \Rightarrow x_3 = 1$

$3x_2 - 1 + 2 \times 3 = -1 \Rightarrow 3x_2 - 1 + 6 + 1 = 0 \Rightarrow 3x_2 = -6$

$x_2 = -2$

$2x_1 - 2 - 1 + 6 = 5 \Rightarrow 2x_1 - 3 + 6 - 5 = 0 \Rightarrow 2x_1 - 2.$

$x_1 = 1, \; x_2 = -2, \; x_3 = 1, \; x_4 = 3.$

EXERCISES

$x_1 + 5x_2 = 7$

$2x_1 - 7x_2 = -5$

Ans $x_1 = -8, \; x_2 = 3$

(2) $4x - 3y + z = -8$

$-2x + y - 3z = -4$ Ans $x = -3, y = 1, z = 2$

$x - y + 2z = 3$

(4) $4x + 8y - 4z = 4$ Ans $x = -3, y = 1 \; z = -2$

$3x + 8y + 5z = -11$

$-2x + y + 12z = -17.$

The Jacobi method

The Jacobi method is describe as given the system of linear equation

$$a_{11}x_1 + a_{12}x_2 + - - - + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + - - - + a_{2n}x_n = b_2.$$
$$\vdots$$
$$a_{mn}x_1 + a_{m2}x_2 + - - - a_{mn}x_n = b_n$$

Next we make $x_1, x_2 - - - x_n$ the subject of the formular in each case as shown below

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - - - a_{1n}x_n\right]$$
$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_2 x_1 - a_{23}x_3 - - - a_{2n}x_n\right].$$
$$\vdots$$
$$x_n = \frac{1}{a_{mn}}\left[b_m - a_{m1}x_1 - a_{m2}x_2 - - a_{mn-1}x_{n-1}\right]$$

If the initial solution is $(x_1^0, x_2^0, - - - x_n^0)$ then a better approximation to the solution is

$$x_1' = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^0 - a_{13}x_3^0 - - - a_n x_n^0\right]$$
$$x_2' = \frac{1}{a_{mn}}\left[b_m - a_{m1}x_1^0 + a_{m2}x_2^0 - - - a_{mn-1}x_{m-1}^0\right]$$

The accuring of the values can be, improved up on my applying the formular recursively, that is

$$x_1^2 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^1 - - - - a_{1n}x_n^1\right].$$

$$x_2^2 = \frac{1}{a_{22}}\left[b_2 - a_{22}x_2^1 - - - - - a_{2n}x_n^1\right].$$

$$x_n^2 = \frac{1}{a_{mn}}\left[b_m - a_{m1}x_1^1 - - - - a_{mn-1}x_{n-1}^1\right]$$

$$x_1^k = \frac{1}{a_{22}}\left[b_2 - a_{21}x_2^{k-1} - - - - a_{2n}x_n^{k-1}\right]$$

$$x_n^k = \frac{1}{a_{mn}}\left[b_m - a_{m1}x_1^{k-1} - - - a_{mn-1}x_{n-1}^{k-1}\right]$$

Note that: the jacobi method is one of the simplest itaractions to implement, while its Convergence make it too slow for use in many problem's, it is worthwide to consider, since it form the basis of other methods

Example

Solve the system using jacobi method

$$20x + y - 2z = 17 \Rightarrow x = \frac{1}{20}\left[17 - y + 2z\right]$$

$$3x + 20y - z = -18 \Rightarrow y = \frac{1}{20}\left[-18 - 3x + z\right]$$

$$2x - 3y + 20z = 25 \Rightarrow z = \frac{1}{20}\left\{25 - 2x + 3y\right]$$

$x = 0.85 - 0.05y + 0.12 \quad z = 0$

$y = -0.9 - 0.15x + 0.05z$

$z = 1.25 - 0.1x + 0.15y$

$x^1 = 0.85 - 0.05(0) + 0.1(0) = 0.85$

$y^1 = -0.9 - 0.15(0) + 0.05(0) = -0.9$

$z^1 = 1.25 - 0.1(0) + 0.15(0) = 1.02$

$x^2 = 0.85 - 0.05(0.9) + 0.9(1.25) = 1.02$

$y^2 = -0.9 - 0.1(0.85) + 0.15(1.25) = 0.965$

$z^2 = 1.25 - 0.1(0.85) + 0.15(-0.9) = 1.03$

$x = 1, \; y = -1, \; z = 1$

Note that after 5th and 6th iteration

$x = 1.0000, \; y = -0.9999$ and $z = 0.9999$

$83x + 11y - 4z = 95$

$7x + 52y + 13z = 104$

$3x + 8y + 29z = 71$

up to three iterations

taking $(x^0, y^0, z^0) = (1, 1, 1)$ as initial approximation

GAUSS SEIDEL METHOD

$$x_1^1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^0 - a_{13}x_3^0 - \cdots - a_{1n}x_n^0\right]$$

$$x_2^1 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^1 - a_{23}x_3^0 - \cdots - a_{2n}x^{k-1}\right]$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^1 - a_{32}x_2^1 - \cdots - a_{3n-1}x^{k-1}\right]$$

$$x_n = \frac{1}{a_{nm}}\left[b_m - a_{m1}x_1' - a_{m2}x_2' - \cdots - a_{nm-1}x_{n-1}'\right]$$

Thus

$$\cancel{x_1^k = \frac{1}{a_{11}}\left[b_1 - a_{m1}x_1' - a_{m2}x_2' - \cdots - a_{nm}\right.}$$

$$x_1^k = \frac{1}{a_{11}}\left[b_1 - a_{12}x^{k-1} - a_{13}x_3^{k-1} - \cdots - a_{1n}x^{k-1}\right]$$

$$x_2^k = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^k - a_{23}x_3^{k-1} - \cdots - a_{2n}x_n^{k-1}\right]$$

$$\vdots$$

$$x_n^k = \frac{1}{a_{mn}}\left[b_m - a_{m1}x_1^k - a_{m2}x_2^k - \cdots - a_{mn-1}x_{n-1}^{k-1}\right]$$

Note ① The Gauss-seidel shene Converges factor than the Jacob's method since newer values of the x's are used once they are avaible

② it is more widely use consequently.

Example

Use the Gauss seidel to solve the system

$20x + y - 2z = 17$ take (x^0, y^0, z^0)

$3x + 20y - z = -18$ $= (1, 1, 1)$ as initial

$2x - 3y + 20z = 25$ approximation.

Solut^n For $k = 1$

$x = \frac{1}{20}[17 - y + 2z]$ $\Rightarrow x = 0.85 - 0.05y + 0.12$

$y = \frac{1}{20}[-18 - 3x + z] \Rightarrow y = -0.9 - 0.15x + 0.052$

$z = \frac{1}{20}[25 - 2x + 3y] \Rightarrow z = 1.25 - 0.1x + 0.15y$

$x^1 = 0.85 - 0.05(0) + 0.1(0) = 0.85$

$y^1 = -0.9 - 0.15(0.85) + 0.05(0) = -1.02$

$z^1 = 1.25 - 0.1(0.85) + 0.15(-1.02) = 1.012.$

For $k=2$

$x^2 = 0.85 - 0.05(-1.02) + 0.1(1.012) = 1.0022$

$y^2 = -0.9 - 0.15(1.0022) + 0.05(1.012) = -0.9997$

$z^2 = 1.25 - 0.1(1.0022) + 0.15(-0.9997) = 0.9998$

Example 2

Use Gauss-seidel method to solve the system

$83x + 11y - 4z = 95$

$7x + 52y + 13z = 104$

$3x + 8y + 29z = 71$

take $(x^0, y^0, z^0) = (1, 1, 1)$

Solut^n

$x = \frac{1}{83}[95 - 11y + 4z] \Rightarrow x = 1.1446 - 0.1325y + 0.0481z$

$y = \frac{1}{52}[104 - 7x - 13z] \Rightarrow y = 2.0000 - 0.1346x - 0.2500z$

$z = \frac{1}{29}\{71 - 3x - 8y] \Rightarrow z = 2.4482 - 0.1034x - 0.2758y$

$x^1 = 1.1446 - 0.1325(1) + 0.0481(1) = 1.0602.$

$y^1 = 2.0000 - 0.1346(1.0602) - 0.2500(1) = 1.6073$

$z^1 = 2.4482 - 0.1034(1.0602) - 0.2758(1.6073) = 1.8952$

$x^2 = 1.0226$

$y^2 = 1.3885$

$z^2 = 1.9555$

EXERCISES

① Use Gauss-Seidel method to evaluate

ⓐ $10x + 2y + z = 9$ take $(x^0, y^0, z^0) = (1,1,1)$.

$2x + 20y - 2z = -44$

$-2x + 3y - 10z = 22$

ⓑ $28x + 4y - z = 32$

$x + 3y + 10z = 24$ take $(x^0, y^0, z^0) = (1,1,1)$.

$2x + 17y + 4z = 35$

② Use the Jacobi's method and Gauss-Seidel method to solve and Compare the solutⁿ to

$10x_1 - 2x_2 - x_3 - x_4 = 3$

$-2x_1 + 10x_2 - x_3 - x_4 = 15$ take $(x_1^0, \cancel{x_2^0}, x_3^0, \cancel{x_4^0})$

$-x_1 - x_2 + 10x_3 - 2x_4 = 27$ $(1, 1, 1, 1)$.

$-x_1 - x_2 - 2x_3 + 10x_4 = -9$

after 3 itration

Assign:

Discuss the numerical solution of higher order differential equation.

Scanned by CamScanner

20-02-2018

Eigenvalues and Eigenvectors

Let $A = (a_{ij})$ be an $n \times n$ matrix, if X is a vec
then a number d is called the eigenvalue of A
In this case, X is called the eigenvectors of A.
The Concept of eigenvalues and the corresponding vectors has a very important applications in science and engineering such as electrical circuit oscillation etc.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Then the eigenvalues can be computed from the determinant $AX - dX = 0$

$$|A - d| = 0$$

That is,

$$\begin{vmatrix} a_{11} - d & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} - d & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = 0$$

The determinant above results in a polynomial equation in d ie $f(d) = P_0 d^n + P_1 d^{n-1} + \cdots + P_n = 0$

The system is called the characteristics equation or Latert equation. The roots of the equat- given the eigenvalues (which may be real or complex) and the corresponding vectors called the eigenvectors.

There are ~~two~~ methods of computing the eigenvalues of a system ① The direct method (Analytical method)

② The iterative or numerical method.

Directs methods (Analytical method).

The direct method include

① Cayley- Hamilton method

② Gaussian method.

Iterative methods.

This methods include

① power method

② Jacobi method.

③ Deflation method.

<u>Example:</u>

Find the eigenvalues and the corresponding eigen-vectors of $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ $\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

<u>Solut^{n}</u>

$$AX - \lambda X = 0 \Rightarrow (A - \lambda)X = 0 \Rightarrow$$

$$|A - \lambda| = 0$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda) - 4 = 0$$

$\Rightarrow 10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$

$\Rightarrow \lambda^2 - \lambda - 6\lambda + 6 = 0$

$\Rightarrow \lambda(\lambda-1) - 6(\lambda-1) = 0$

$(\lambda-1)(\lambda-6) = 0 \quad \Rightarrow \lambda = 1$ or $\lambda = 6$.

$Ax - \lambda x = 0$

$(A - \lambda)x = 0$ for $\lambda = 1$

$$\Rightarrow \begin{pmatrix} 5-1 & 4 \\ 1 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$4x_1 + 4x_2 = 0$

$x_1 + x_2 = 0$

$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

let $x_1 = 1$, $x_2 = -1$

for $\lambda = 1$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

for $\lambda = 6$

$(A - \lambda)x = 0$

$$\begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$-x_1 + 4x_2 = 0 \qquad\qquad x_1 = 4x_2$

$x_1 - 4x_2 = 0 \quad \Rightarrow \quad x_2 = \frac{x_1}{4} \Rightarrow x_1 = 1, x_2 = \frac{1}{4}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/4 \end{pmatrix}.$$

$\therefore$ The eigenvalues are $\lambda = 1$ and $\lambda = 6$ and the corresponding eigenvectors are $(1, -1)$ and $(1, 1/4)$

Iterative ~~methods~~ or numerical methods.

Sometimes the methods discussed ealier may ~~be~~ not be applicable to some other systems. In that case the iterative method(s) sought for these methods include:

1. Power method, Jacobi method and deflation method etc

Power method

This method is used to compute the dominant eigenvalue and the corresponding eigenvectors

<u>Definition</u> if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of an $n \times n$ square matrix, then an eigenvalue λ_k is said to be the dominant eigen value of A if the absolute value of λ_k is greater then the absolute value of the other eigenvalues ie $|\lambda_k| > |\lambda_i|$ for $i = 1, 2, 3 \ldots n$, $i \neq k$.

Example

1. $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ have the eigenvalues $\lambda = 1$ and 6 therefore the dominant eigvalue is 6.

② $A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 6 \\ -1 & -2 & -1 \end{pmatrix}$

have, $\lambda_1 = 0$, $\lambda_2 = -4$, $\lambda_3 = 3$

$\therefore$ The dominant eigenvalue is $|-4| = 4$

③ $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ have $\lambda_1 = -2$, $\lambda_2 = 2$ hence the system has no dominant eigenvalue because $|-2| = |2| = 2$.

The power method is due to the German mathematician Richard Von mises (1853 — 1953)

Suppose an $n \times n$ matrix A has a dominant eigenvalue, if X_0 is an initial approximate to the eigenvector, then

$$X = AX_0 = \lambda_1 X_1$$
$$X^2 = AX_1 = \lambda_2 X_2$$
$$X^3 = AX_2 = \lambda_3 X_3$$
$$X^n = AX_{n-1} = \lambda_n X_n$$

λ_n is the dominant eigen value
X^n is the corresponding eigen vector

Example

If $A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$ let $X_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be the initial approximation, find the dominant eigenvalue

Solutⁿ:-

$$X' = AX_0 = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \overset{\lambda_1}{6}\overset{X_1}{\begin{pmatrix} 1 \\ 0.33 \end{pmatrix}}$$

$$X^2 = AX_1 = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0.33 \end{pmatrix} = \begin{pmatrix} 4.66 \\ 2.67 \end{pmatrix} = \overset{\lambda}{4.66}\begin{pmatrix} 1 \\ 0.57 \end{pmatrix}$$

$$X^3 = AX_2 = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0.57 \end{pmatrix} = \begin{pmatrix} 5.14 \\ 2.67 \end{pmatrix} = \overset{\lambda_3}{5.14}\begin{pmatrix} 1 \\ 0.48 \end{pmatrix}^{X_3}$$

$$X^4 = AX_3 = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0.48 \end{pmatrix} = \begin{pmatrix} 4.96 \\ 2.52 \end{pmatrix} = \overset{\lambda_4}{4.96}\begin{pmatrix} 1 \\ 0.51 \end{pmatrix}^{X_4}$$

$$X^5 = AX_4 = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0.51 \end{pmatrix} = \begin{pmatrix} 5.02 \\ 2.49 \end{pmatrix} = 5.02\begin{pmatrix} 1 \\ 0.50 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$$

$$|A - \lambda| = 0 \Rightarrow \begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(1+\lambda) + 6 = 0$$

$$4 + 4\lambda - \lambda - \lambda^2 + 6$$

$$-\lambda^2 + 3\lambda + 10 = 0 \Rightarrow \lambda^2 - 3\lambda - 10 = 0$$

$$\lambda^2 - 5\lambda + 2\lambda - 10 = 0$$

$$\lambda(\lambda-5) + 2(\lambda-5) \Rightarrow (\lambda-5)(\lambda+2) = 0$$

$$\lambda = 5,\ \lambda = -2$$

$$AX - \lambda X = 0 \Rightarrow (A-\lambda)X = 0$$

$$\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$-x_1 + 2x_2 = 0 \qquad x_2 = \frac{x_1}{2}$$

$$3x_1 - 6x_2 = 0 \qquad x_1 = 1$$

$$x_1 = 2x_2 \qquad x_2 = \frac{1}{2}$$

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① $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$X_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

② $A = \begin{pmatrix} 5 & -6 & 1 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}$

$X_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

③ $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$

$X_0 = (1, 1, 0)^T$

06-03-2018

Inverse of matrices

let A be a square matrix (invertible) a matrix is called the inverse of A denoted by A^{-1} iff $AA^{-1} = A^{-1}A = I$, where I is the Identity matrix. The inverse of a matrix A Can be determined by several methods.

Direct (Analytical) method

The direct or analytical method of finding the inverse of a matrix include

i) The adjoint method (ie $A^{-1} = \frac{\text{adj} A}{|A|}$

Numerical method

This method include

1. Doolittle method
2. Choleski's method
3. Inversion method

Inversion method

If A is a matrix whose approximate inverse B

Then $AB = I + E$

$E = AB - I$

$(AB)^{-1} = (I + E)^{-1}$

$B^{-1}A^{-1} = I - E + E^2 - E^3 + \cdots$

$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)x^{n-2}y^2}{2!} - \cdots$

$(I+E)^{-1} = (I - E + E^2) - E^3 + \cdots$

$B^{-1}A^{-1} = I - E + E^2$

multiply by B.

$BB^{-1}A^{-1} = B(I - E + E^2)$

$A^{-1} = B(I - E + E^2)$.

Example

$A = \begin{pmatrix} 5 & 2 \\ 3 & -1 \end{pmatrix}$ if $B = \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & -0.4 \end{pmatrix}$ is an approximate inverse of A

using $A^{-1} = B(I - E + E^2)$

where $E = AB - I$

$$E = \begin{pmatrix} 5 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & -0.4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$E = \begin{bmatrix} 1.1 & 0.2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0 \end{bmatrix}$$

$$E^2 = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.01 & 0.02 \\ 0 & 0 \end{bmatrix}$$

$$A^{-1} = B\left(I - E + E^2 \right]$$

$$= \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & -0.4 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.1 & 0.2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0.01 & 0.02 \\ 0 & 0 \end{pmatrix} \right]$$

$$A^{-1} = \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & -0.4 \end{pmatrix} \begin{pmatrix} 0.91 & -0.18 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 0.091 & 0.182 \\ 0.273 & -0.454 \end{pmatrix}$$

EXERCICES

find the approximate inverse of A

(1) $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{pmatrix} -2.45 & 2.05 \\ 1.45 & -0.95 \end{pmatrix}$

(ii) $A = \begin{bmatrix} 1 & 10 & 1 \\ 2 & 0 & 1 \\ 3 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0.4 & 2.14 & -1.4 \\ 0.14 & 0.14 & -0.4 \\ -0.85 & -3.8 & 2.8 \end{bmatrix}$

Doolittle method

* A matrix is said to be triangular matrix if the element above or below the main diagonal are zero

* A matrix is said to be lower triangular matrix if the element above the main diagonal are zero

example

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

* A matrix is said to be an upper triangular matrix. if the elements below the main diagonal are zero eg

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{pmatrix}$$

Doolittle method :- This method assumes that every matrix A can be expressed as a product of two triangular matrix that is $A = LU$.

ie $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$

where U is an upper triangular matrix

and L is a lower triangular matrix in this case U^{-1} can be determined from the relation $U^{-1} = B$.

$\Rightarrow UB = I$

Similarly if $L^{-1} = Y$.

Then $LY = I$

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$L = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix}$

$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$

$A = LU$

$a_{11} = u_{11}$	$a_{21} = L_{21} u_{11}$	$a_{31} = L_{31} u_{11}$
$a_{12} = u_{12}$	$a_{22} = L_{21} u_{12} + u_{22}$	$a_{32} = L_{31} u_{12} + L_{32} u_{22}$
$a_{13} = u_{13}$	$a_{23} = L_{21} u_{13} + u_{23}$	$a_{33} = L_{31} u_{13} + L_{32} u_{33} + u_{33}$

Example

Find the inverse of the matrix A

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ -1 & 2 & 1 \end{pmatrix}$$ and hence solve the equation

$3x + y + 2z = 6$

$2x - 3y - z = 5$

$-x + 2y + z = 1$

Solution

Using $A = LU$

$$\begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ -1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

$U_{11} = 3$

$U_{12} = 1$

$U_{13} = 2$

$L_{21} U_{11} = 2$

$L_{21} (3) = 2$

$L_{21} = 2/3$

$L_{21} U_{12} + U_{22} = -3$

$2/3 (1) + U_{22} = -3$

$U_{22} = -3 - \frac{2}{3} = -11/3$

$U_{22} = -11/3$

$L_{21} U_{13} + U_{23} = -1$

$\frac{2}{3}(1) + U_{23} = -1$

$U_{23} = -1 - \frac{4}{3} = -7/3$

$U_{23} = -7/3$

$L_{31} U_{11} = -1$

$L_{31}(3) = -1$

$L_{31} = -1/3$

$L_{31} U_{12} + L_{32} U_{22} = 2$

$-\frac{1}{3}(1) - \frac{11}{3} L_{32} = 2$

$-\frac{11}{3} L_{32} = 2 + \frac{1}{3}$

$L_{32} = -7/3 * 3/11 = -7/11$

$L_{32} = -7/11$

$$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 1$$

$$-\frac{1}{3}(2) + (-7/11)(-7/3) + U_{33} = 1$$

$$-\frac{2}{3} + \frac{49}{33} + U_{33} = 1$$

$$U_{33} = 1 + \frac{2}{3} + \frac{49}{33}$$

$$U_{33} = \frac{33+22-49}{33} = 2/11$$

$$U_{33} = \frac{2}{11}$$

Now

$$L = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & -7/11 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 0 & -\frac{11}{3} & -7/3 \\ 0 & 0 & 2/11 \end{pmatrix}$$

Let $L^{-1} = Y$ where

$$Y = \begin{pmatrix} Y_{11} & 0 & 0 \\ Y_{21} & Y_{22} & 0 \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix}$$

$$LY = I$$

$$LY = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -1/3 & -7/11 & 1 \end{pmatrix} \begin{pmatrix} Y_{11} & 0 & 0 \\ Y_{21} & Y_{22} & 0 \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$y_{11} = 1$

$\frac{2}{3} y_{11} + y_{21} = 0$

$y_{21} = -\frac{2}{3}$

$y_{22} = 1$

$-\frac{1}{3} y_{11} - \frac{7}{3} y_{21} + y_{31} = 0$

$-\frac{11}{33} + \frac{14}{33} + y_{31} = 0$

$y_{31} = -\frac{1}{11}$

$-\frac{7}{11} y_{22} + y_{32} = 0$

$y_{32} = \frac{7}{11}$

$y_{33} = 1$

$$L^{-1} = Y = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{1}{11} & 7/11 & 1 \end{pmatrix}$$

$$U^{-1} = Z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ 0 & z_{22} & z_{23} \\ 0 & 0 & z_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 1/11 & -\frac{5}{2} \\ 0 & -3/11 & -7/2 \\ 0 & 0 & 11/2 \end{pmatrix}$$

$A = LU$

$$A^{-1} = U^{-1}L^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{11} & -5/2 \\ 0 & -3/11 & -7/2 \\ 0 & 0 & 11/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{1}{11} & 7/11 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & -3/2 & -5/2 \\ \frac{1}{2} & -5/2 & -7/2 \\ -1/2 & 7/2 & 1/2 \end{pmatrix}$$

Solve the equation $\quad B = \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}$

$A^{-1} A X = A^{-1} B$

$X = A^{-1} B$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ -13 \\ 20 \end{pmatrix}$$

27-03-2018

Choleski's method

If A is a symmetric matrix, then $A = LL^T$

(Note: that if a matrix A is symmetric then $A = A^T$)

L - Lower triangular matrix and L^T is the transpose of L.

$$A^{-1} = (LL^T)^{-1} = (L^T)^{-1}L^{-1} = (L^{-1})^T L^{-1}$$

If
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}}_{L}$$

$$\begin{bmatrix} L_{11} & L_{21} & L_{33} \\ 0 & L_{22} & L_{32} \\ 0 & 0 & L_{33} \end{bmatrix} = L^T$$

$a_{11} = L_{11}^2$, $L_{11} = \sqrt{a_{11}}$

$a_{13} = L_{11}L_{31}$

$L_{11}L_{21} = a_{12}$

$L_{31} = \dfrac{a_{13}}{\sqrt{a_{11}}}$

$L_{21} = \dfrac{a_{12}}{\sqrt{a_{11}}}$

Example A^{-1} where $A = \begin{pmatrix} 4 & 2 & 12 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{pmatrix}$

Solution

Using $A = LL^T$

$4x_1 + 2x_2 + 14x_3 = 14$

$2x_1 + 17x_2 - 5x_3 = -101$

$14x_1 - 5x_2 + 83x_3 = 155$

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$$\begin{pmatrix}4&2&14\\2&17&-5\\14&-5&83\end{pmatrix} = \begin{bmatrix}L_{11}&0&0\\L_{21}&L_{22}&0\\L_{31}&L_{32}&L_{33}\end{bmatrix}\begin{bmatrix}L_{11}&L_{21}&L_{31}\\0&L_{22}&L_{3}\\0&0&L\end{bmatrix}$$

$L_{11}^2 = 4$ $\quad \boxed{L_{11}=2}$ $\qquad L_{21}L_{11} = 2 \quad L_{21}=1$

$L_{11}L_{21} = 2, \boxed{L_{21}=1}$ $\qquad L_{21}^2 + L_{22}^2 = 17, \boxed{L_{22}=4}$

$L_{11}L_{31} = 14, \boxed{L_{31}=7}$ $\qquad L_{21}L_{31} + L_{22}L_{32} =$

$7 + 4L_{32} = -5, \boxed{L_{32}=-}$

$L_{31}^2 + L_{32}^2 + L_{33}^2 = 83.$

$49 + 9 + L_{33}^2 = 83 \Rightarrow L_{33} = 5$

$$L = \begin{pmatrix}L_{11}&0&0\\L_{21}&L_{22}&0\\L_{31}&L_{32}&L_{33}\end{pmatrix} = \begin{pmatrix}2&0&0\\1&4&0\\7&-3&5\end{pmatrix}$$

Let $L^{-1} = Y = \begin{bmatrix}y_{11}&0&0\\y_{21}&y_{22}&0\\y_{31}&y_{32}&y_{33}\end{bmatrix}$

$LL^{-1} = I$

$LY = I$

$$\begin{pmatrix}2&0&0\\1&4&0\\7&-3&5\end{pmatrix}\begin{pmatrix}y_{11}&0&0\\y_{21}&y_{22}&0\\y_{31}&y_{32}&y_{33}\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$2y_{11} = 1 \Rightarrow \boxed{y_{11} = 1/2}$

$y_{11} + 4y_{21} = 0$

$4y_{21} = -\frac{1}{2} \Rightarrow \boxed{y_{21} = -\frac{1}{8}}$

$4y_{22} = 1$

$\boxed{y_{22} = \frac{1}{4}}$

$7y_{11} - 3y_{21} + 5y_{31} = 0$:

$\frac{7}{2} + \frac{3}{8} + 5y_{31} = 0 \Rightarrow \frac{31}{8} + 5y_{31} = 0$

$\boxed{y_{31} = -\frac{31}{40}}$

$-3y_{22} + 5y_{32} = 0$

$-3/4 + 5y_{32} = 0$

$\boxed{y_{32} = 3/20}$ $\quad y_{33} = \frac{1}{5}$

$$L^{-1} = Y = \begin{pmatrix} 1/2 & -1/8 & -\frac{31}{40} \\ 0 & \frac{1}{4} & 3/20 \\ 0 & 0 & 1/5 \end{pmatrix}$$

but $A^{-1} = (L^{-1})^T L^{-1}$

$$(L^{-1})^T = \begin{pmatrix} 1/2 & 0 & 0 \\ -\frac{1}{8} & 1/4 & 0 \\ -\frac{31}{40} & 3/20 & 1/5 \end{pmatrix}$$

$A^{-1} = (L^{-1})^T L^{-1}$

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1/8 & 1/4 & 0 \\ -\frac{31}{40} & 3/20 & 1/5 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{8} & -\frac{31}{40} \\ 0 & 1/4 & 3/20 \\ 0 & 0 & 1/5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1/4 & -\frac{1}{16} & -\frac{31}{80} \\ -\frac{1}{16} & 5/64 & \\ & & \end{pmatrix} \qquad B = \begin{pmatrix} 14 \\ -101 \\ 155 \end{pmatrix}$$

$Ax = B \Rightarrow X = A^{-1}B$